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Can dark matter in galaxies be explained by relativistic corrections?

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Abstract
Cooperstock and Tieu proposed a model of galaxy, based on ordinary GR, in which no exotic dark matter is needed to explain the flat rotation curves in galaxies. I will present the arguments against this model. In particular, I will show that in their model the gravitational field is generated not only by the ordinary matter distribution, but by a infinitely thin, massive and rotating disc as well. This is a serious and incurable flaw and makes the Cooperstock–Tieu metric unphysical as a galaxy model.

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1. Introduction

The dark matter problem is one of the most important issues in astrophysics. It has been observed in various scales and objects since 1960s and has motivated a lot of research in various directions. On the galactic scale the dark matter problem amounts to the discrepancy between the observed velocity curves on one hand and the visible matter distribution on the other. It seems difficult to reconcile both without adding a large, massive dark component, possibly made of exotic matter, in the form of a halo. Other attempts involve various modifications of laws of gravity at galactic scales.

The most conservative approach to the problem so far has been suggested in [1, 12, 13]. The authors propose to explain the discrepancy by general relativistic corrections to the motion of stars in galaxies, without exotic dark matter and with standard GR as the gravity theory.

The paper is organized as follows: in the following section I will briefly present the Cooperstock–Tieu model of galaxy. Then I will point out the main problems of this model and prove that it harbours an additional layer of mass in the midplane. Finally I will discuss other criticisms of the Cooperstock and Tieu's paper model and their replies.

2. The Cooperstock–Tieu model

Cooperstock and Tieu [1] consider a gravitating, pressureless cloud of gas of density ρ and 4-velocity u^μ. They assume the spacetime to be axisymmetric, stationary and asymptotically flat. The Einstein equations in this case read

$$G_{\mu\nu}[g_{\alpha\beta}] = \frac{8\pi G}{c^2}\rho u_\mu u_\nu. \tag{1}$$

Following van Stockum and Bonnor [3, 4], Cooperstock and Tieu assume that the dust flow follows a geodesic field which is simultaneously a Killing vector. Consequently, the metric element in appropriately adapted coordinates reads

$$\mathrm{d}s^2 = -\mathrm{e}^\nu(u\,\mathrm{d}z^2 + \mathrm{d}r^2) - r^2\,\mathrm{d}\varphi^2 + (c\,\mathrm{d}t - N\mathrm{d}\varphi)^2. \tag{2}$$

The authors perform now a formal perturbative expansion of (1) in $G^{1/2}$ and obtain the following set of equations:

$$u = 1 \tag{3}$$

$$N_{rr} + N_{zz} - \frac{N_r}{r} = 0 \tag{4}$$

$$\frac{N_z^2 + N_r^2}{r^2} = \frac{8\pi G\rho}{c^2}. \tag{5}$$

Finally they propose a solution to (3)–(5) in the form of a sum

$$N(r,z) = r\frac{\partial}{\partial r}\sum_n C_n\,\mathrm{e}^{-k_n|z|}J_0(k_n r) \tag{6}$$

with arbitrary C_n's. The rotation curves $v(r,z)$ can easily be read out from the function N,

$$v(r,z) = \frac{Nc}{r} \tag{7}$$

and so can be the mass distribution via (5). For any sum of finite number of terms the mass density falls off exponentially with z and r, so the galaxy is localized and has a finite total mass.

The model is very simplified, as the rotation is rigid, contrary to what we observe in nature. It nevertheless has an interesting feature: if rotation curves of ordinary galaxies are fitted to (6), it turns out that the total mass of the galaxy needed is reduced an order of magnitude compared to Newtonian calculations [1].

Obviously there are also several problems with this model. Firstly, its dynamics is very different from Newtonian, while the motion and conditions in galaxies seem to be non-relativistic. In fact the model does not seem to have a Newtonian limit even when $GM \to 0$: every particle of the dust evolves around the z-axis along a circular orbit. This is simply not possible in the Newtonian theory unless we are dealing with an infinitely long, rotating column of gas or with a perfectly flat disc.

Secondly, solution (6) contains a non-differentiable function. While such solutions are not mathematically excluded, one has to treat them with care. They may arise if we choose a coordinate system with a singularity or if the source functions are distributional. We will see that in the Cooperstock–Tieu model the latter is true.

3. Perturbation expansion revisited

Expanding in terms of a dimensional parameter is only a formal mathematical procedure which may or may not yield a good approximate solution. One gains much more physical insight from an expansion in terms of a dimensionless parameter. In the problem of the galactic motion we can do in the following way [2, 6].

First, we introduce rescaled coordinates $\tilde{x}^\mu = L^{-1}x^\mu$, dimensionless mass density $\tilde{\rho} = L^3M^{-1}\rho$ and dimensionless velocity $\tilde{u}^\mu = c^{-1}u^\mu$ (L is the characteristic length of the galaxy, M its total mass and c the light speed). The Einstein equations become now

$$G_{\mu\nu}[\tilde{g}_{\alpha\beta}] = \lambda\tilde{\rho}\tilde{u}_\mu\tilde{u}_\nu \tag{8}$$

with dimensionless $\lambda = 8\pi GML^{-1}c^{-2}$. In the Galaxy this quantity is of the order of 10^{-6} which makes the perturbative expansion around the flat metric a viable method.

We now take $g = \eta + \lambda^{1/2}g^{(1)} + \lambda g^{(2)} + \cdots$ and $u = u^{(0)} + \lambda^{1/2}u^{(1)} + \cdots$ and apply the harmonic (De Donder) gauge to fix the coordinate system. When applied to (8), it yields at the two lowest orders

$$\Delta g^{(1)}_{\mu\nu} = 0 \tag{9}$$

$$\Delta g^{(2)}_{\mu\nu} = F(g^{(1)}, u^{(0)}). \tag{10}$$

The first equation is the ordinary, flat Laplace equation and does not have any asymptotically flat (localized) non-zero solutions. This sharply contradicts the results of Cooperstock and Tieu who claimed to have presented a non-trivial solution which is asymptotically flat. The reason of the contradiction turns out to be that (6) does not really solve (3)–(5) in the sense that it has a delta function rather than 0 on the right hand side of (4). This can be interpreted as an additional source of gravitational field in the form of a disc.

4. Singular disc of matter in the Cooperstock–Tieu model

I will now prove the existence of the singular disc in (2). Recall that if K is a Killing vector, τ is its dual 1-form, the following identity holds,

$$\mathrm{d}\star\mathrm{d}\tau = \tfrac{1}{3}R^{\mu\alpha}\tau_\alpha\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma, \tag{11}$$

which integrated over an arbitrary three-dimensional domain V yields

$$\int_{\partial V}\star\mathrm{d}\tau = \int_V \mathrm{d}\star\mathrm{d}\tau = \frac{4\pi G}{3}\int_V (2T^{\alpha\mu}\tau_\alpha - T\tau^\mu)\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma. \tag{12}$$

The integral over the boundary of V is called the Komar integral and is an analogue of the Gauss law in electromagnetism: a *surface* integral is equal to a *volume* integral of some part of the stress–energy tensor, which is the source of the gravitational field.

In the Cooperstock–Tieu spacetime take V to be a cylinder given by $0 < r < R, -a < z < a$ and K to be the timelike Killing vector $\frac{\partial}{\partial t}$. The Komar integral has now the interpretation of mass generating the gravitational field.

Keep the radius R constant while shrinking the cylinder's height to zero ($a \to 0$). If T consisted solely of dust, with no singular component, the limit of the right-hand side of the Komar integral would be zero, as the volume shrinks to 0. However it is not the case in CT metric:

$$\int_{\partial V}\star\mathrm{d}K = 4\pi\int_0^R \mathrm{d}r\,\frac{N}{r}\frac{\partial N}{\partial z}\bigg|_{z=a}$$

$$\to -4\pi\int_0^R \mathrm{d}r\, r\left(\sum_n C_n k_n J_0'(k_n r)\right)\left(\sum_m C_m k_m^2 J_0'(k_m r)\right) \neq 0. \tag{13}$$

The same argument goes for the axial Killing vector $\frac{\partial}{\partial\varphi}$ [2]. The reason can be traced back to the presence of $|z|$ function in (6).

This proves the existence of an additional gravitating component (disc) at $z = 0$ which has both mass and angular momentum. If we evaluate all components of the singular part of $T_{\mu\nu}$, it turns out that the 'matter' of the disc is quite exotic [5].

This problem cannot be cured by smoothing out $|z|$ in (6), because any smoothened function would necessarily violate equation (9). This perturbative argument has actually been strengthened by a fully non-perturbative one: it was proved analytically that all van Stockum metrics are either non-asymptotically flat, or possess singularities of negative mass [8].

5. A review of other criticisms of the Cooperstock–Tieu model

The model presented in [1] was criticized on various grounds by several other authors. Cooperstock and Tieu posted later two papers addressing the issues raised by the author of this paper and others [12, 13]. In this section 1, will present a summary of arguments and results of the opponents as well as the replies of Cooperstock and Tieu. I will also include a brief commentary.

Garfinkle [10] argues that the post-Newtonian approximation should work very well in the context of galactic dynamics. Therefore if the dark matter problem arises in the Newtonian theory, so it does in GR. A galaxy model which differs significantly from the Newtonian would have to involve a geon, i.e. quasi-stationary vacuum configuration of strong gravitational field. However, in view of the theorem of Klainerman and Christodoulou [11], it is very unlikely that such models exist: a weak-field configuration would disperse quickly, while in a strong one the motion of the stars would not be slow comparing to c, as we observe in Nature.

Cooperstock and Tieu reply that they do not impose the harmonic gauge condition in their paper, as was done in this paper or [10]. But this is obviously irrelevant in a general covariant theory like GR. A result established in the harmonic coordinates system must be valid in any other.

A similar line of reasoning is followed by Cross in [6], where he notes that the rotation of the dust assumed by Coopertock and Tieu is necessarily rigid (covariant shear vanishes while the rotation does not). He performs carefully the expansion of the Einstein equations in the terms of $\sqrt{\lambda}$ finding out that the coupling between the angular momentum and the metric is too weak to affect the rotation curves in a significant way and produce large deviations from the Newtonian approximation.

With respect to the problem of $z = 0$ singularity, Cooperstock and Tieu claim it is merely a mathematical issue. While admitting that there is a singularity in their metric, they exclude the $z = 0$ plane from the domain of integration of the Komar integral (13), and therefore from the spacetime. They claim the physical matter density is given by the limit of the continuous distribution as $z \to 0$.

However, this kind of excision is simply wrong. The Komar integral plays the role of the Gauss law in general relativity, i.e. it relates certain surface integrals with the volume integral of some of the components of the gravitational field source. What makes it particularly useful in this situation is the fact that it can easily capture any distributional or singular sources of gravitational field, which one might omit when paying attention only to the continuous part of the matter distribution (i.e. the dust).

Consider a simple example in electrostatics: imagine an electrically charged plane embedded in a continuous distribution of charge. If one simply excluded the plane from the domain of integration of the Gauss law, by excising a volume involving the plane and shrinking it to zero, like in [13], the resulting total charge would omit the charge of the plane.

It is then incorrect to define the charge density at the plane as the limit of the continuous charge density, because the charge density is a distribution rather than a continuous function. Therefore it is the appropriate *integral law* which enables us to calculate its true value at $z = 0$ rather than the limit $z \to 0$.

Cooperstock and Tieu note further that a layer of negative mass would repulse any positive mass nearby and therefore alter the geodesic motion in the spacetime. This remark is of course correct—and in fact the layer of negative mass is needed exactly for that.

As was already pointed out, in the Newtonian gravity it is simply impossible to create a stable configuration of ordinary matter rotating around the z-axis and of finite thickness. Any particle off the $z = 0$ plane would suffer attraction towards that plane, simply due to the imbalance of total masses above and below. Therefore it could not follow an azymuthal circular trajectory all the time, which requires the z component of acceleration to vanish. To escape that argument one has to either consider an infinitely thin disc, an infinitely thick column or add something repulsing to the middle. Theorems of [8, 9] suggest that the situation in GR is quite similar: when considering pressureless, rigidly rotating dust, one must either give up asymptotic flatness, or add a repulsing singularity to the metric. It is exactly their repulsion which makes the whole system stable.

Rigorous, analytical results concerning the van Stockum class of metrics have been established in [8] and [9]. In [8] it is proven that the metrics are either non-asymptotically flat, or contain singularities of negative mass. This is in good agreement with the hand-waving argument of the previous paragraph. The reply of Cooperstock and Tieu in [13] is obviously incorrect: they maintain that the analytical results do not hold for their approximate solution. But if the approximate metric is supposed to model a physical galaxy in accordance with GR, it must be understood as an approximation of the correct one and every negative result concerning the correct metric should automatically invalidate the whole model.

The results of [9] are even stronger: global solutions with circularly rotating dust cannot be extended beyond some distance $R_{\max}$ from the z-axis. One might try to match them with asymptotically flat vacuum solutions, but this turns out to be impossible without singularities. Once again, the arguments are based on the analysis of the full non-linear equations.

In [14] the authors propose a different model of a galaxy as a stationary configuration of the van Stockum type. Their solution suffers from exactly the same problems as the original Cooperstock–Tieu problems. Namely, it harbours a singularity, though this time located along the z-axis. A singularity of this type (or a singular layer of matter) seems to be even more exotic than the dark matter halo.

Finally, I should mention one paper [7] which deals with the original Cooperstock–Tieu model from the observational point of view. The authors point out that the resulting local matter density, the vertical density profile and local integrated surface density near the Sun are inconsistent with observations.

Summarizing, the Cooperstock–Tieu metric or its modifications like [14] are not good galaxy models and do not solve the dark matter problem in astronomy and astrophysics.

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